Topology Vol. 12, pp. 229–232. Pergamon Press, 1973. Printed in Great Britain

AN ALGEBRAIC FIBRE BUNDLE OVER $\mathbb{P}_1$ THAT IS NOT A VECTOR BUNDLE†

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(*Received* 15 *September* 1972)

OUR bundle Z is a locally trivial algebraic fibre bundle with fibre 2-dimensional affine space and a global section, over the projective line $\mathbb{P}_1$ (over a field k). It is not isomorphic—as an algebraic fibre bundle over $\mathbb{P}_1$—to a vector or affine bundle. G. Hamrick and D. Sullivan pointed out that the theory of microbundles shows any such locally trivial bundle is diffeomorphic to a vector bundle if characteristic $k = 0$. We thank A. Mattuck, under whose direction the thesis containing this example was written, our colleague G. Hamrick for pointing out the need for Lemma 4, and N. Greenleaf and R. Speiser for helpful discussions of these results.

The example arises from a study of ideals in $k[[x, y]]$, the ring of power series in two variables over an algebraically closed field k [2]. Let m be the maximal ideal (x, y) of $k[[x, y]]$. An ideal I is *linear* if $m^2 \not\supset I$. I has colength n if $\dim_k k[[x, y]]/I = n$. Note that $k[[x, y]] = \prod_0^\infty A_j$ where $A_j \cong m^j/m^{j+1}$ are the homogeneous polynomials in x, y of degree j.

Z is the variety parametrizing linear ideals of colength 4. Such an ideal may be normalized so as to have one of the following forms

$$I = (y + c_0 x + c_1 x^2 + c_2 x^3, m^4) \qquad c_i \in k$$

OR (1)

$$I = (x + b_0 y + b_1 y^2 + b_2 y^3, m^4) \qquad b_i \in k.$$

Here the c_i (or the b_i) may take on arbitrary values in k. The graded linear ideals of colength 4 have the form (l, m^4), where $l \in A_1$ is a linear form, determined up to constant multiple; so these are parametrized by the projective line $\mathbb{P}_1$. The section $i: \mathbb{P}_1 \to Z$ comes from the inclusion of graded ideals in all ideals, and the projection $\pi : Z \to \mathbb{P}_1$ corresponds to the map taking an ideal to its associated graded ideal. Thus, if $I = (y + c_0 x + c_1 x^2 + c_2 x^3, m^4)$, $\pi(I)$ is the ideal $(y + c_0 x, m^4)$. Clearly, $\pi \circ i =$ identity.

THEOREM 1. *Z is locally trivial over $\mathbb{P}_1$, but is not a vector bundle.*

To prove this, we must consider the variety Z' parametrizing linear ideals of colength 3. These ideals have the form

$$I = (y + c_0 x + c_1 x^2, m^3) \qquad c_i \in k$$

OR (2)

$$I = (x + b_0 y + b_1 y^2, m^3) \qquad b_i \in k$$

† This result is part of a Ph.D. thesis written under the direction of A. Mattuck at M.I.T. in 1970.

‡ Partially supported by NSF GP 29028.

The graded linear ideals of colength 3 have the form (l, m^3) where $l \in A_1$ and therefore are parametrized by $\mathbb{P}_1$. We let i', π' be the inclusion and projection: $\mathbb{P}_1 \underset{i'}{\overset{\pi'}{\leftrightarrows}} Z'$. There is a morphism

$$p: Z \to Z'$$

corresponding to the map taking a linear ideal of colength 4 to the ideal $I + m^3$ of colength 3. Clearly p is surjective, $p \circ i = i'$ and $p \circ i \circ \pi = i' \circ \pi' \circ p$. We need four lemmas:

LEMMA 1. *Z' is the line bundle $\mathcal{O}(3)$ over $\mathbb{P}_1$.*

LEMMA 2. *$Z'' = p^{-1}(i'(\mathbb{P}_1))$ is the line bundle $\mathcal{O}(4)$ over $\mathbb{P}_1$.*

LEMMA 3. *There are no sections $s: Z' \to Z$ of p.*

LEMMA 4. *If $\pi: Z \to \mathbb{P}_1$ is a vector bundle, p is a vector bundle homomorphism.*

These suffice to prove Theorem 1, since, if Z were a vector bundle, Lemmas 1, 2 and 4 would imply there is an exact sequence of vector bundles over $\mathbb{P}_1$:

$$0 \to \mathcal{O}(4) \to Z \xrightarrow{p} \mathcal{O}(3) \to 0. \tag{3}$$

Since degree $\mathcal{O}(4) >$ degree $\mathcal{O}(3)$, Grothendieck [1] shows $Z = \mathcal{O}(4) \oplus \mathcal{O}(3)$. Then there is a section s of p, contradicting Lemma 3. So Z cannot be a vector bundle.

We now prove the Lemmas. Let U_x be the open set in $\mathbb{P}_1$ where $x \neq 0$ and U_y the open set where $y \neq 0$. The ideals I having the first form of (1) correspond to the closed points of $\pi^{-1}(U_x)$ in Z. That c_1, c_2 are arbitrary in (1) shows that Z is trivial over U_x. We now investigate the transition functions of Z from $\pi^{-1}(U_x)$ to $\pi^{-1}(U_y)$, and of Z' from $(\pi')^{-1}$ $(U_x$ to $(\pi')^{-1}(U_y)$.

Proof of Lemma 1. Suppose a linear ideal I of colength $n \geq 3$ has both forms

Let

$$I = (y + c_0x + \cdots + c_{n-2}x^{n-1}, m^n) = (x + b_0y + \cdots + b_{n-2}y^{n-1}, m^n).$$

$$g = x + b_0y + \cdots + b_{n-2}y^{n-1}.$$

$$C = c_0 + \cdots + c_{n-2}x^{n-2} \in k[[x]].$$

Then

$$y \equiv -Cx \bmod I$$

$$y^i \equiv (-1)^i C^i x^i \bmod I.$$

$$g \equiv x + \sum_{0}^{n-2} (-1)^{i+1} b_i C^{i+1} x^{i+1} \in I. \tag{4}$$

But $I \cap k[[x]] \subset (x^n)$ else I contains $y + c_0x + \cdots + c_{n-2}x^{n-1}$ and x^{n-1}, so has colength $\leq$ colength $(y + c_0x + \cdots + c_{n-2}x^{n-1}, x^{n-1}) = n - 1$. Hence we conclude the polynomial in the right side of (4) is 0 mod (x^n). Thus, from its coefficients on x, x^2, and x^3 we conclude

$$1 = b_0c_0$$

$$b_1c_0{}^2 = b_0c_1$$

$$b_2c_0{}^3 - 2b_1c_0c_1 + b_0c_2 = 0 \quad \text{if } n > 3$$

We may rewrite these

$$\begin{aligned} b_0 &= 1/c_0 \\ b_1 &= b_0{}^3 c_1 \\ b_2 &= b_0{}^3(2b_1 c_0 c_1 - b_0 c_2) \quad \text{if } n > 3 \\ &= 2b_0{}^{-1} b_1{}^2 - b_0{}^4 c_2. \end{aligned} \tag{5}$$

The first equation of (5) is the transition function from U_x to U_y. The first two equations of (5) are the transition functions for the bundle $\mathcal{O}(3)$ over $\mathbb{P}_1$, and taking $n = 3$ we conclude Lemma 1.

Proof of Lemma 2. The closed points of $p^{-1}(i'(\mathbb{P}_1))$ correspond to those ideals of (1) such that

$$b_1 = 0 \quad \text{OR} \quad c_1 = 0.$$

Taking $n \leq 4$ in (5), we see that the transition function for such ideals over $U_x \cap U_y$ are

$$\begin{aligned} b_0 &= 1/c_0 \\ b_1 &= c_1 = 0 \\ b_2 &= -b_0{}^4 c_2 \quad (\text{since } b_1 = 0) \end{aligned}$$

therefore $p^{-1}(i'(\mathbb{P}_1))$ is the bundle $\mathcal{O}(4)$ over $\mathbb{P}_1$, which shows Lemma 2.

Proof of Lemma 3. Suppose $s: c_2 = h(c_1, c_0)$ is a section $Z' \to Z$ of p over U_x. Here h is a polynomial in c_1 and c_0. From (5) we conclude that over $U_x \cap U_y$:

$$b_2 = 2b_0{}^{-1} b_1{}^2 - b_0{}^4 h(b_0{}^{-3} b_1, b_0{}^{-1}). \tag{6}$$

No term of $b_0{}^4 h(b_0{}^{-3} b_1, b_0{}^{-1})$ can cancel $2b_0{}^{-1} b_1{}^2$ in (6), and therefore it is not possible to continue the section s over all U_y (where b_0 can be 0). This completes the proof of Lemma 3.

Proof of Lemma 4. Since the tangent bundle to Z at the "0-section" $i(\mathbb{P}_1)$ is $\mathcal{O}(4) \oplus \mathcal{O}(3)$, it is the only vector bundle that could be isomorphic to Z. Suppose f is a vector bundle isomorphism

$$f: \theta(4) \oplus \theta(3) \to Z.$$

To prove Lemma 4 it suffices to show $p \circ f$ is a vector bundle homomorphism. First, since there is only the 0 fibred morphism $\mathcal{O}(4) \to Z' \simeq \mathcal{O}(3)$.

$$p \circ f(\mathcal{O}(4)) = 0 \quad (\text{meaning the image of } p \circ f \text{ is } i'(\mathbb{P}_1)) \tag{7}$$

$\therefore f(\mathcal{O}(4)) \subset Z'' \cong \mathcal{O}(4)$ by Lemma 2.

But the only fibre bundle monomorphisms, $\mathcal{O}(4) \to \mathcal{O}(4)$ are vector bundle homomorphisms, multiplication by a non-zero constant, so

$$f \mid \mathcal{O}(4): \mathcal{O}(4) \to Z'' \text{ is a vector bundle isomorphism.} \tag{8}$$

Second, suppose u is a section of $\mathcal{O}(3)$; then

$$p \circ f \mid \mathcal{O}(4) + u = p \circ f(u), \text{ since} \tag{9}$$

$p \circ f \mid (\mathcal{O}(4) + u) - p \circ f(u)$ is a fibre bundle map from $\mathcal{O}(4)$ into $\mathcal{O}(3)$, so is 0.

Thus, by (7) and (9), to show $p \circ f$ is a vector bundle homomorphism it suffices to show

$$p \circ f \mid \mathcal{O}(3) \to Z' \text{ is a vector bundle homomorphism}$$

But the only fibre bundle morphisms $\mathcal{O}(3) \to Z' \cong \mathcal{O}(3)$ are vector bundle homomorphisms, multiplication by a constant. This completes the proof of Lemma 4, and of Theorem 1.

THEOREM 2. *If $n \geq 4$, the variety Z_n of dimension $n - 1$ parametrizing linear ideals of colength n is locally trivial over $\mathbb{P}_1$, but it is not a vector bundle.*

Proof (Sketch). $n = 4$ is Theorem 1. Suppose $n \geq 5$ and assume inductively that Z_{n-1} is not a vector bundle. Let p_n be the projection $p_n: Z_n \to Z_{n-1}$ corresponding to the map of ideals $I \to I + m^{n-1}$. Then, p_n is surjective and, we may show as in Lemma 2 that

$$p_n^{-1} \text{ (graded ideals)} \cong \mathcal{O}(n)$$

and that if Z_n is a vector bundle, $\mathcal{O}(n)$ is a sub-bundle; hence Z_{n-1} would be the vector bundle $Z_n/\mathcal{O}(n)$, contradicting the induction assumption.

If we take $k = \mathbb{C}$ the varieties Z_n are holomorphic complex manifolds over complex $\mathbb{P}_1$. We may also take $k =$ Reals and obtain locally trivial real analytic manifolds over the real projective line. Z_n parametrizes the right cosets of the triangular subgroup in the automorphism group of A/m^{n+1}. (See [2].)

The Z_n are particular cases of a whole class of such locally trivial but in general non-vector bundles over flag-like manifolds arising naturally in ideal theory and described in [2]. Each such bundle Z_T in [2] parametrizes the ideals in $k[[x, y]]$ of a given type T of finite length and is locally trivial over the complete variety G_T parametrizing graded ideals of the same type. Here the type T of I is the infinite sequence of integers $T(I) = (t_0, \ldots, t_j, \ldots,)$ where $t_j = \dim_k A_j/I_j$ and I_j is $(I \cap m^j + m^{j+1})/m^{j+1}$. The length of $T = \sum_0^\infty t_j =$ colength I.

Z_n corresponds to the type $T: t_j = 1$ if $0 \leq j \leq n - 1$, $t_j = 0$ if $j \geq n$.

There are many more locally trivial non-vector bundles. In [4] we classify those over $\mathbb{P}_1$ that, like Z_n, have a sequence

$$Y = Y_n \to Y_{n-1} \to \cdots \to Y_2 = \mathbb{P}_1$$

of "quotient" locally trivial bundles, such that $Y_i \to Y_{i-1}$ has "kernel" (in the sense Z'' is a "kernel" of p) a positive line bundle L_i and $L_i > L_{i-1}$. Such bundles with specified $\{L_i\}$ are parametrized by an infinite union of affine spaces.

We have shown $Z_4 \neq \mathcal{O}(4) \oplus \mathcal{O}(3)$ only as fibre bundles over $\mathbb{P}_1$. We conjecture that Z_4 and $\mathcal{O}(4) \oplus \mathcal{O}(3)$ are algebraically non-isomorphic 3-folds which are topologically the same.

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